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Convective instability of stably stratified water in the ocean

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A recent theoretical description of interactions between surface waves and currents in the ocean is extended to allow density stratification. The interaction leads to a convective instability even when the density stratification is statically stable. An unspecified random surface wave field is permitted provided that it is statistically stationary.

The instability can be traced to torques produced by variations of a 'vortex force'. Non-diffusive instabilities produced by this mechanism in water of infinite depth are explored in detail for arbitrary distributions of the destabilizing force. Stability is determined by an eigenvalue problem formally identical to that determining normal modes of infinitesimal internal waves in fluid with a density profile that is not monotone and thereby has a statically unstable region. Some tentative remarks are offered about the problem when dissipation is allowed.

Application of the present theory to Langmuir circulations is discussed. Also, according to the present theory, internal wave propagation should be modified by the vortex force arising from the interaction between the surface waves and the current.

1. Introduction

In recent papers by Craik & Leibovich (1976, hereafter referred to as CL) and by Leibovich (1977, hereafter referred to as I) a set of equations was developed that describes the interaction of surface waves with a mean current driven ultimately by the wind stress. The equations are valid provided that the wave particle speeds are large compared with the mean current and provided that the time scale for formation of the current is long compared with the period of a typical surface wave.

The procedure, most completely described in I, involves two-timing followed by averaging over a time span long compared with a wave period (the fast time) but short compared with the time scale of formation of the wind-drift current. The details of the surface wave motion are suppressed, and their rectified effect on the current formation appears through a term formally identical to the Stokes drift.

A cellular roll motion in the upper water layers, resembling Langmuir circulations (see Pollard (1976) for the latest review of the Langmuir-circulation phenomenon), arises from the rectified equations when the Stokes drift possesses spanwise variability. This might, in principle at least, arise from a bimodal surface wave directional spectrum. Langmuir-like motions, arising in this way as a direct wave–current interaction described by the rectified equations, were studied extensively for homogeneous bodies of water in CL, in I and in Leibovich & Radhakrishnan (1977). The existence of spanwise variability in the Stokes drift, even for a bimodal wave directional spectrum, requires that the waves be locked in phase for many (perhaps 100 or more) wave periods. This, as pointed out by C. J. R. Garrett (private communication), is unlikely actually to occur except in restricted circumstances. Preliminary evidence obtained by O. M. Phillips (private communication) supports Garrett's view.

On the other hand, Craik (1977) has pointed out that Langmuir-like motions are described by the rectified equations for a completely random surface wave field which is characterized by a wave drift *uniform* in the spanwise direction. As observed in I, the rectified equations are mathematically nearly analogous to the equations of thermal convection: this remains true even if the wave drift is uniform in the spanwise direction. Craik (1977) exploited this fact to show that an instability analogous to thermal convection occurs in homogeneous water, and has illustrated the point by examples.

The present paper derives a generalized form of the rectified equations of I, based upon the Boussinesq approximation, that allows the water to be density stratified, and then employs these equations to treat the inviscid stability of a time-independent current to infinitesimal roll disturbances. Currents with sufficiently large shear are found to be unstable to such disturbances, even when the water is stably stratified.

The joint effect of the Stokes drift and a sheared current is equivalent to an alteration of the density profile near the surface. Therefore, if no instability occurs, the effectively modified density profile will alter the propagation characteristics of internal waves.

Let the speed of the Stokes drift depend upon depth alone, be denoted by $U_s(z)$, and be parallel to the basic current $U(z)$. Let the water be density stratified with density $\bar{\rho}(z)$. Then our principal result states that inviscid instability occurs when the function

$$\mathscr{M}(z) = \frac{dU_s}{dz}\frac{dU}{dz} + \frac{g}{\rho_r}\frac{d\bar{\rho}}{dz}$$

is positive in some interval. Here $-z$ is the depth below the mean free water surface and ρ_r is a constant reference density. The condition $\mathscr{M}(z) > 0$ is a criterion of Richardson-number type. Since the maximum of $\mathscr{M}$ typically occurs at the surface $z = 0$, this criterion may be stated in terms of a minimum wind stress τ_w required to cause instability:

$$\tau_w > -\nu_T g \frac{d\bar{\rho}(0)}{dz} \bigg/ \rho_r U_s'(0).$$

Substitution of typical values for the parameters appearing in this criterion indicates that the slightest breeze is destabilizing. In practice, therefore, buoyancy is not effective at suppressing instability. On the other hand, a study of the behaviour of the eigenfunctions shows that buoyancy determines the effective depth of the destabilized layer.

The maximum growth rate for an unstable configuration is

$$\sigma_{\max} = \mathscr{M}^{\frac{1}{2}}(0).$$

Growth-rate estimates from this expression for conditions that typically occur in the ocean are consistent with observed time scales of formation of Langmuir circulations, and the possible application of this theory to the Langmuir-circulation phenomenon is discussed in the final section of the paper.

The eigenvalue problem governing the stability problem is formally identical to that determining the normal modes of infinitesimal internal waves in water of infinite depth. In contrast to the usual considerations of internal wave propagation however, $\mathscr{M}(z)$ is not everywhere positive, and the usual proof of the variation of the discrete spectrum with the wavenumber k must be modified. General forms for $\mathscr{M}(z)$ are considered, except that we require $\mathscr{M}$ to decrease with depth. We show that the growth rate $\sigma(k)$ increases monotonically, taking all values from $\sigma = 0$ at $k = 0$ to $\sigma_{\max}$ as $k \to \infty$. Furthermore the $\sigma(k)$ curve is concave towards the k axis. When $\bar{\rho}$ is asymptotically stable and linear as $z \to -\infty$, the problem possesses a continuous spectrum corresponding to internal waves propagating to or from great depths.

If one is willing to assume a Stokes drift that is linear with depth (a bad assumption), then the linear stability problem, including diffusive effects, is formally analogous to the problem of thermal convection. This analogy is brought out in §5, where the physical basis of the destabilizing effect of the Stokes drift is also explained in terms of torques arising from a 'vortex force'. By invoking the assumption of a linear Stokes drift, one is able to carry over known results from the literature of thermal convection. The problem closest to the one in this paper is the cooling from above of a deep layer of fluid that is stably stratified. This problem has been treated by Whitehead & Chen (1970). We may therefore quote their results for the critical (effective, in this case) Rayleigh numbers and most unstable wavenumbers for steady convection of infinitesimal amplitude, provided that we bear in mind the assumption that has been made. This is also carried out in §5.

2. Equations for wind-driven convective mixing of stratified water

The procedure described in I will be followed, but density stratification will be allowed for. The basic idea in I is to average the Navier–Stokes equations over time intervals large compared with the period of the dominant surface waves but small compared with the time required for secondary currents to develop. The resulting equations can be used to study motions whose characteristic time scale is large compared with a surface wave period and whose characteristic velocities are small compared with the orbital speed of water particles in the dominant surface waves, although the applicability of further assumptions (e.g. use of a constant eddy viscosity) must be evaluated.

In order to save space, the derivation here will lean heavily on I. In particular, we shall not make the problem explicitly dimensionless to identify the small parameters, but shall depend upon familiarity with the procedure in I, which underlies the present development. For example, although we shall not formally introduce two time scales, we are thinking of the same two-time analysis as was presented in I.

The Boussinesq approximation will be made, the density being written as

$$\rho_r + \bar{\rho}(z) + \rho(\mathbf{x}, t),$$

where ρ_r is a constant reference density, $\bar{\rho} + \rho(\mathbf{x}, t)$ represents the deviation of ρ from ρ_r, and $\bar{\rho}$ is the density deviation at the initial time $t = 0$. The vorticity equations (ignoring the earth's rotation) according to Phillips (1966, p. 18) are

$$\left.\begin{aligned} \frac{D\boldsymbol{\omega}}{Dt} &= \boldsymbol{\omega} . \nabla \mathbf{q} - \frac{g}{\rho_r} \operatorname{curl} \rho \mathbf{k} + \nu \nabla^2 \boldsymbol{\omega}, \\ \boldsymbol{\omega} &= \operatorname{curl} \mathbf{q}, \end{aligned}\right\} \tag{1}$$

where $\mathbf{k}$ is a unit vector in the z direction (taken vertically upwards from the mean free water surface).

Let the velocity field $\mathbf{q}$ be written as the superposition of an irrotational surface wave field $\mathbf{u}_w$, with wave slope proportional to the small parameter ϵ, and a still smaller contribution $\mathbf{v}$, proportional to $\epsilon\delta c$, where c is the phase speed of the surface waves and δ is a second small parameter that accounts for all other motions:

$$\mathbf{q} = \mathbf{u}_w + \mathbf{v}. \tag{2}$$

The parameter δ is explicitly identified in terms of the applied wind stress and other given parameters of the problem in I. In particular, $\epsilon\delta$ is a measure of the vorticity in the current field.

If the resulting vorticity equations are 'processed' (by perturbation and averaging in time) as in I, and if the buoyancy term $(g/\rho_r)\,\mathrm{curl}\,(\rho\mathbf{k})$ is assumed not to be large compared with $\boldsymbol{\omega}.\nabla\mathbf{v}$, then the method of averaging used in I carries over directly and results in the (dimensional) vorticity equations

$$\langle\boldsymbol{\omega}\rangle = \mathrm{curl}\,\langle\mathbf{v}\rangle,$$
$$\frac{\partial\langle\boldsymbol{\omega}\rangle}{\partial t} + (\langle\mathbf{v}\rangle + \mathbf{u}_s).\nabla\langle\boldsymbol{\omega}\rangle = \langle\boldsymbol{\omega}\rangle.\nabla(\langle\mathbf{v}\rangle + \mathbf{u}_s) - \frac{g}{\rho_r}\,\mathrm{curl}\,(\langle\rho\rangle\mathbf{k}) + \nu_T\nabla^2\langle\boldsymbol{\omega}\rangle, \tag{3}$$

where for any scalar or vector function f

$$\langle f\rangle = \frac{1}{T}\int_{-T}^{T} f\,dt.$$

Here T is an averaging time long compared with the surface wave period but short compared with the time required for the development of currents, and the vector $\mathbf{u}_s$ is identical to the Stokes drift:

$$\mathbf{u}_s = \left\langle \int^t \mathbf{u}_w dt . \nabla\,\mathbf{u}_w \right\rangle \tag{4}$$

(Phillips 1966, p. 31). Also, ν has been replaced by an eddy viscosity ν_T.

An appropriate measure of $(g/\rho_r)\,\mathrm{curl}\,(\rho\mathbf{k})$ is given in terms of the prescribed function $\bar{\rho}$ by $(g/\rho_r)\,\partial\bar{\rho}/\partial z$, so that the assumptions made about this term may be stated as

$$-\frac{g}{\rho_r}\frac{\partial\bar{\rho}}{\partial z} = O(|\langle\boldsymbol{\omega}\rangle.\nabla\langle\mathbf{v}\rangle|).$$

The smallest length scale that can be treated by the averaged equations is of order κ^{-1}, where κ is the wavenumber of the dominant surface waves, so we must have

$$-\frac{g\,\partial\bar{\rho}}{\rho_r\partial z} = O(\kappa^2 v^2), \tag{5}$$

where v is $O(\epsilon\,\delta c)$. The left-hand side of (5) is the square of the Brunt–Väisälä frequency $N(z)$, so validity of the procedure requires that the dimensionless parameter

$$(\delta\epsilon)^{-2}(N/\sigma_s)^2 \tag{6}$$

should not be large. Here σ_s is the frequency of the surface waves. A typical value for $\delta\epsilon$ would be 0·01, so the equations should be valid for motions in which the ratio N/σ_s is not large compared with 0·01. For typical values of N, this is satisfied by all except fairly long surface waves.

Equation (3) can be integrated to give

$$\frac{\partial\langle \mathbf{v}\rangle}{\partial t}+\langle \mathbf{v}\rangle . \nabla\langle \mathbf{v}\rangle = \mathbf{u}_s \times \langle \boldsymbol{\omega}\rangle - \nabla\pi - \frac{g}{\rho_r}\langle\rho\rangle\mathbf{k}+\nu_T\nabla^2\langle \mathbf{v}\rangle. \tag{7}$$

The term π includes the mean kinetic energy of the wave motion in addition to the averaged pressure.

One of the principal questions of interest motivating this research is the mixing accomplished by organized convective motions. To treat this question, it is convenient to adopt the customary step taken in thermal convection, and replace density by temperature in the Boussinesq approximation. Although we shall not treat any details of mixing here, we shall set forth the equations in a form suitable for that purpose. Thus, if β is the coefficient of thermal expansion, we write

$$\bar{\rho} = -\beta \bar{T}(z)\rho_r, \quad \rho = -\rho_r\beta\theta,$$

where the temperature is

$$T(\mathbf{x},t) = T_r+\bar{T}(z)+\theta(\mathbf{x},t). \tag{8}$$

The temperature is governed by the energy equation, which is (with the usual approximations of thermal convection)

$$\theta_t+\mathbf{q}.\nabla\theta+\mathbf{k}.\mathbf{q}\,\bar{T}'(z) = \alpha_T\nabla^2\theta, \tag{9}$$

where the prime denotes d/dz and α_T is the (turbulent) thermal diffusivity. It is convenient for the moment to replace α_T by ν_T/Pr_T, where Pr_T is a turbulent Prandtl number. If we decompose $\mathbf{q}$ into wave and current contributions as in (2), recall that $\mathbf{u}_w = O(\epsilon) \gg \mathbf{v}$, and assume that (as in I) the inverse Reynolds number $\kappa\nu_T/\epsilon c$ is sufficiently small (and Pr_T is not too small) and that $\alpha_T\kappa/c = O(\epsilon^2)$ then θ may be expanded in a series in ϵ,

$$\theta = \theta_0+\epsilon\theta_1+\ \ldots,$$

and the first two coefficients are determined by the equations

$$\partial\theta_0/\partial t = 0, \quad \partial\theta_1/\partial t = -\mathbf{k}.\mathbf{u}_w\bar{T}'(z).$$

The first equation asserts that θ_0 does not change on the fast time scale (comparable to a wave period), although it may vary on a longer time scale. The second equation may be integrated:

$$\theta_1 = -\bar{T}'(z)\int^t \mathbf{k}.\mathbf{u}_w dt+\langle\theta_1\rangle,$$

where $\langle\theta_1\rangle$ is the mean with respect to the fast time and may vary on the slow time scale.

Divide the temperature disturbance θ into its mean and fluctuating parts: $\theta = \langle\theta\rangle+\theta'$. To this order

$$\langle\theta\rangle = \theta_0+\langle\theta_1\rangle, \quad \theta' = -\bar{T}'(z)\int^t \mathbf{k}.\mathbf{u}_w dt.$$

We substitute this decomposition into the energy equation and average over the fast time:

$$\frac{\partial\langle\theta\rangle}{\partial t}+\langle(\mathbf{u}_w+\mathbf{v})\rangle.\nabla\langle\theta\rangle-\left\langle(\mathbf{u}_w+\mathbf{v}).\nabla\left(\bar{T}'(z)\int^t \mathbf{k}.\mathbf{u}_w dt\right)\right\rangle$$
$$+\langle(\mathbf{k}.\mathbf{v}+\mathbf{k}.\mathbf{u}_w)\rangle\bar{T}'(z) = \alpha_T\nabla^2\langle\theta\rangle.$$

But, since $\langle \mathbf{u}_w \rangle = 0$,

$$\langle \mathbf{u}_w + \mathbf{v} \rangle . \nabla \langle \theta \rangle = \langle \mathbf{v} \rangle . \nabla \langle \theta \rangle,$$

$$\langle \mathbf{k}.\mathbf{v} + \mathbf{k}.\mathbf{u}_w \rangle \overline{T}'(z) = \mathbf{k}.\langle \mathbf{v} \rangle \overline{T}'(z)$$

and

$$\left\langle (\mathbf{u}_w + \mathbf{v}).\nabla \left(\overline{T}'(z) \int^t \mathbf{k}.\mathbf{u}_w dt \right) \right\rangle = \left\langle \mathbf{u}_w . \nabla \left(\overline{T}'(z) \int^t \mathbf{k}.\mathbf{u}_w dt \right) \right\rangle.$$

The last term is the correlation between the fluctuating wave field and the wave-induced temperature fluctuations. It may be shown to vanish under fairly general circumstances. To see this, rewrite the term as

$$\left\langle \mathbf{u}_w . \nabla \overline{T}' \int^t \mathbf{k}.\mathbf{u}_w dt \right\rangle = \left\langle \frac{\partial}{\partial t} \left\{ \tfrac{1}{2} \overline{T}'' \left(\int^t \mathbf{k}.\mathbf{u}_w dt \right)^2 + \overline{T}' \int^t \mathbf{u}_w dt . \nabla \int^t \mathbf{k}.\mathbf{u}_w dt \right\} \right\rangle$$
$$- \overline{T}' \left\langle \int^t \mathbf{u}_w dt . \nabla (\mathbf{k}.\mathbf{u}_w) \right\rangle. \qquad (10)$$

The term in the curly bracket is differentiated with respect to t, so that its time average vanishes, while the last term is [cf. (4)] $-\overline{T}'(z)\mathbf{k}.\mathbf{u}_s$. The vertical component of the Stokes drift is assumed to vanish, so the entire expression on the left-hand side of (10) vanishes. This leaves the 'ordinary' energy equation (where $\langle w \rangle = \mathbf{k}.\langle \mathbf{v} \rangle$)

$$\partial \langle \theta \rangle / \partial t + \langle \mathbf{v} \rangle . \nabla \langle \theta \rangle + \langle w \rangle \overline{T}'(z) = \alpha_T \nabla^2 \langle \theta \rangle. \qquad (11)$$

The full set of equations for the developing currents may now be collected:

$$\partial \langle \mathbf{v} \rangle / \partial t + \langle \mathbf{v} \rangle . \nabla \langle \mathbf{v} \rangle = \mathbf{u}_s \times \langle \boldsymbol{\omega} \rangle - \nabla \pi - \beta g \langle \theta \rangle \mathbf{k} + \nu_T \nabla^2 \langle \mathbf{v} \rangle, \qquad (12a)$$

$$\partial \langle \theta \rangle / \partial t + \langle \mathbf{v} \rangle . \nabla \langle \theta \rangle + \langle w \rangle \overline{T}'(z) = \alpha_T \nabla^2 \langle \theta \rangle, \qquad (12b)$$

$$\nabla . \langle \mathbf{v} \rangle = 0. \qquad (12c)$$

We note that the Stokes drift u_s (for waves in deep water) vanishes exponentially fast with depth, so the equations reduce to the unconditioned Boussinesq approximation to the Navier–Stokes equations at depths of the order of the wavelength of the surface waves. For stably stratified waters, therefore, the set reduces to the one conventionally used to describe internal wave propagation when depths at which $u_s \ll 1$ are reached.

At the mean free surface $z = 0$, the vertical current vanishes:

$$\langle w(x, y, 0, t) \rangle = 0. \qquad (13a)$$

As in I, the problem of motion developing under the action of an applied wind stress (in the x direction, say) may be treated by imposing the stress boundary conditions

$$\nu_T(\partial \langle u \rangle / \partial z) = u_*^2, \quad \nu_T(\partial \langle v \rangle / \partial z) = 0, \qquad (13b,c)$$

where u_* is the water friction velocity associated with the applied stress and use has been made of (13a). For water of infinite depth, all disturbances should decay as $z \to -\infty$.

Finally, thermal boundary and initial conditions must also be imposed on the mean temperature $\langle \theta \rangle$. By virtue of the definition of $\overline{T}$, at $t = 0$

$$\langle \theta \rangle (\mathbf{x}, 0) = 0. \qquad (13d)$$

If we assume that the temperature at infinite depths is held fixed, then

$$\langle\theta\rangle \to 0 \quad \text{as} \quad z \to -\infty. \tag{13e}$$

At the free surface $z = 0$, any suitable boundary condition on

$$T_r + \overline{T}(z) + \langle\theta\rangle$$

may be imposed. For example, one may take a prescribed temperature $T_0(x,y,t)$, in which case

$$\langle\theta\rangle(x,y,0,t) = T_0(x,y,t) - T_r - \overline{T}(0),$$

or a prescribed heat flux T_0', in which case

$$\frac{\partial}{\partial z}\langle\theta\rangle(x,y,0,t) = T_0' - \frac{\partial\overline{T}}{\partial z}(0).$$

Here

$$T_0' = \frac{\partial T}{\partial z}(x,y,0,t).$$

3. Linearized instability of a shear flow

We now assume that the action of the wind stress has produced a time-independent parallel shear flow

$$\langle\mathbf{v}\rangle = U(z)\mathbf{i} \tag{14}$$

and that surface waves have produced a Stokes drift current that varies with depth only:

$$\mathbf{u}_s = U_s(z)\mathbf{i}, \tag{15}$$

with $dU/dz \geqslant 0$ and $dU_s/dz \geqslant 0$ for all z. In the absence of a stable density gradient, the situation represented by (14) and (15) may be expected to be unstable to infinitesimal disturbances because of their close similarity to the equations of thermal convection (see §4) and this has been confirmed by Craik (1977). (Craik also applied the results of Foster (1965, 1968) to consider the stability of the problem above when viscosity is included and U evolves in time as a solution of the Rayleigh problem with an imposed surface stress.)

We consider the inviscid linearized stability of (14) and (15) to roll disturbances in the presence of a stable density gradient

$$d\bar{\rho}/dz = -\rho_r \beta\, d\overline{T}/dz \leqslant 0. \tag{16}$$

If diffusion is ignored arbitrary functions U, $\overline{T}$ and U_s satisfy (12).

If the state described by $U(z)$ and $\overline{T}(z)$ is disturbed by the perturbations (u,v,w) and θ, i.e.

$$\langle\mathbf{v}\rangle = (U(z)+u, v, w), \quad T - T_r = \overline{T}(z) + \theta,$$

where $(u,v,w) \ll U$ and $\theta \ll \overline{T}$, then the linearized forms of (12) may be reduced to the following three equations for u, w and θ:

$$\Delta[w_t + (U+U_s)\,w_x] = \Delta_1(\beta g\theta - U_s' u) + [w_x(2U' + U_s')]_z + \nu_T \Delta^2 w, \tag{17a}$$

$$\Delta[u_t + (U+U_s)\,u_x + wU'] = -\beta g\theta_{xz} + [w_x(2U' + U_s')]_x + (U_s' u_x)_z + \nu_T \Delta^2 u, \tag{17b}$$

$$\theta_t + U\theta_x + w\overline{T}' = \alpha_T \Delta\theta, \tag{17c}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial x^2}, \quad \Delta_1 = \Delta - \frac{\partial^2}{\partial z^2}.$$

If we consider only disturbance rolls aligned with the mean flow, then the motion is independent of x and (17) simplifies to

$$\Delta w_t = \Delta_1(\beta g\theta - U_s' u) + \nu_T \Delta^2 w, \tag{18a}$$

$$u_t + wU' = \nu_T \Delta u, \quad \theta_t + w\overline{T}' = \alpha_T \Delta\theta. \tag{18b, c}$$

From the form of (18) the roles of u and θ are seen to be similar, and the analogy with thermal convection is apparent.

If, on the other hand, x-dependent disturbances are allowed, but only non-conducting inviscid flows are considered, the equations may be reduced to a single second-order equation. If we assume the exponential behaviour

$$w = \phi(z)\exp[\sigma t + i(ky + mx)], \tag{19a}$$

$$u = (k^2+m^2)^{-1}(imw_z - k^2 wU'f^{-1}), \tag{19b}$$

$$v = k(k^2+m^2)^{-1}(iw_z + mwU'f^{-1}), \tag{19c}$$

$$\theta = -w\overline{T}'(\sigma + imU)^{-1}, \tag{19d}$$

$$f \equiv \sigma + im(U+U_s) \tag{19e}$$

and assume $\alpha_T = \nu_T = 0$, then $\phi(z)$ satisfies the equation

$$\phi'' - (k^2+m^2)\phi + \left\{\frac{mU''}{if} + \frac{k^2U'U_s'}{f^2} - \frac{(k^2+m^2)N^2}{f(\sigma+imU)}\right\}\phi = 0, \tag{20a}$$

where

$$N^2(z) = \beta g\overline{T}'(z) \tag{20b}$$

is the Brunt–Väisälä frequency. In the absence of surface wave motion, $U_s = 0$ and (20) reduces to the equation governing the stability of inviscid density-stratified flows [cf. Drazin & Howard 1966, equation (3.12)].

In the present paper, we shall confine attention to perturbations consisting of rolls aligned with the mean flow ($m = 0$), for which (20) takes the very simple form

$$\sigma^2(\phi'' - k^2\phi) + k^2\mathcal{M}(z)\phi = 0, \tag{21a}$$

$$\mathcal{M}(z) = U_s' U' - N^2. \tag{21b}$$

The boundary conditions on w require

$$\phi(0) = \phi(-\infty) = 0. \tag{21c}$$

We note that (19b, d) now imply that u and θ also vanish at $z = 0$, and that this is a consequence of the non-diffusive approximation. In applying the equations of § 2 to Langmuir circulations therefore, where u is known to achieve its maximum at the surface, one must include the effects of viscosity (and probably heat conduction) to account properly for the observed surface behaviour. Nevertheless, the growing vertical motion will probably be adequately described by the inviscid problem for sufficiently small times.

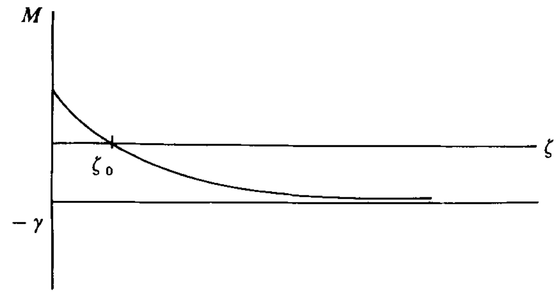

FIGURE 1. Sketch of the form of $M(\zeta)$ considered.

The basic state $(U, \overline{T})$ is unstable if σ^2 is positive and stable if σ^2 is negative. Notice that, if $U_s = 0$ and σ^2 is negative, the problem (21) is identical to that for infinitesimal amplitude internal waves. Thus (21) may also describe internal waves, and shows that the classical Brunt–Väisälä frequency is modified near the surface where surface wave activity is significant.

3.1. *Assumptions about* $\mathscr{M}(z)$

Typically U_s' and U' are positive and so is $\overline{T}'$ if the water is stably stratified by density. In addition, as $z \to -\infty$, U_s' and U' both vanish, and the decay of U_s is exponential. We assume that, as $z \to -\infty$, the density stratification is ultimately linear (although this assumption is not required for most of our results, it is convenient), so

$$\beta g \overline{T}' \sim \gamma \quad \text{as} \quad z \to -\infty,$$

where γ is a non-negative constant. Either $\mathscr{M}(z) < 0$ for all z (when $\overline{T}'$ is sufficiently large) or $\mathscr{M}(z)$ is positive for a region $0 > z > -\zeta_0$ and negative for $z < -\zeta_0$. For simplicity, we assume that there is at most one zero of the function $\mathscr{M}(z)$, at $z = -\zeta_0$, and that $\mathscr{M}$ is monotonic decreasing. To avoid writing minus signs frequently, we let

$$z = -\zeta, \quad \mathscr{M}(z) = \mathscr{M}(-\zeta) = M(\zeta).$$

The typical form of M under consideration is therefore shown in figure 1, where $M(0)$ may be either positive or negative. In terms of the independent variable ζ the problem (21) becomes

$$\phi_{\zeta\zeta} + (\lambda M(\zeta) - k^2)\phi = 0, \tag{22a}$$

$$\phi(0) = \phi(\infty) = 0, \quad \lambda \equiv k^2\sigma^{-2}. \tag{22b, c}$$

Equations (22) pose a singular (because of the semi-infinite domain) Sturm–Liouville problem. The problem possesses both a discrete spectrum ($\lambda > 0$) and a continuous spectrum ($\lambda < 0$), which will be discussed below. The occurrence of a sign change in $M(\zeta)$ renders some standard results concerning the spectrum inapplicable, but, as will be shown, these results can be obtained by a modified proof.

3.2. σ^2 *is real*

The wavenumber k is specified, and σ^2 is determined as the eigenvalue of the system (22). We may refer to Yih (1974) or to Ince (1956, §10.71) for a demonstration that the eigenvalues σ^2 are real, even if $M(\zeta)$ has a change of sign. (Yih's proof is for internal waves, but is directly applicable to this problem as well.) Thus σ is either real, in which case the motion is unstable (since a positive root exists for σ), or is purely imaginary, in which case the motion is oscillatory (a modified internal wave). Unstable motions therefore correspond to $\sigma^2 > 0$ and oscillatory motions to $\sigma^2 < 0$. Note that, since λ is real, we may take the eigenfunctions ϕ to be real.

3.3. *Necessary condition for instability: minimum wind stress for mixing*

Where $M(\zeta) < 0$, the solutions of (22*a*) have $\phi_{\zeta\zeta}$ and ϕ of the same sign, a condition described by Morse & Feshbach (1953, p. 723) as 'exponential behavior'. When $M(\zeta) > 0$, ϕ and $\phi_{\zeta\zeta}$ have opposite signs, or 'sinusoidal behavior' according to Morse & Feshbach. It is clear that, if $M(\zeta) \leqslant 0$ for all ζ, then there are no bounded solutions to (22) except $\phi \equiv 0$ for $\lambda > 0$.

By hypothesis, the maximum value of $M(\zeta)$ occurs at $\zeta = 0$. Therefore a necessary condition for instability is

$$M(0) = U_s'(0)\, U'(0) - \beta g \overline{T}'(0) > 0. \tag{23}$$

It will be shown (§3.7) that this is also a sufficient condition for instability. Since the applied wind stress is related to $U'(0)$ by the condition

$$\rho \nu_T U'(0) = \rho u_*^2 = \tau_w$$

condition (23) implies that the wind stress required to overturn a stable density structure is

$$\tau_w \geqslant \rho \nu_T \beta g \overline{T}'(0)/U_s'(0). \tag{24}$$

3.4. σ^2 *increases with* k^2 *at a decreasing rate*

This is the analogue of a standard result concerning the dispersion relation for ordinary internal waves. The proof given by Yih (1974, p. 274) is valid only for positive $M(\zeta)$. The following proof applies when M is allowed to change sign.

We are interested in the variation of σ^2 with k^2 for a particular mode in the discrete spectrum (i.e. a mode characterized by the number of zeros of its eigenfunction). Consider the eigenfunction $\phi(\zeta; k^2)$ corresponding to the eigenvalue $\lambda(k^2)$. Differentiate (22*a*, *b*) with respect to k^2. If we use the notation

$$\chi(\zeta; k^2) = \partial\phi(\zeta; k^2)/\partial k^2$$

then the differentiation leads to the following problem for χ:

$$\chi_{\zeta\zeta} + (\lambda M(\zeta) - k^2)\chi = \phi - (d\lambda/dk^2)\, M(\zeta)\, \phi, \tag{25a}$$

$$\chi(0) = \chi(\infty) = 0. \tag{25b}$$

Since λ is the eigenvalue corresponding to k^2, the problem (25) has no solution unless the solvability condition

$$\int_0^\infty \left[\phi - \frac{d\lambda}{dk^2} M(\zeta)\phi\right] \phi\, d\zeta = 0 \tag{26}$$

is satisfied.

From the definition of λ ($= k^2/\sigma^2$)

$$d\lambda/dk^2 = k^{-2}(\lambda - \lambda^2 d(\sigma^2)/dk^2).$$

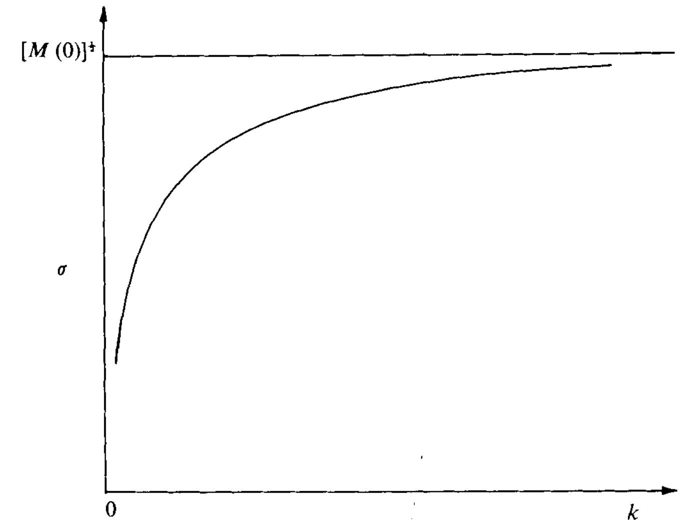

FIGURE 2. General shape of the behaviour of the growth rate as a function of wavenumber: $\sigma(k)$ is concave towards the k axis, zero at $k = 0$, and approaches $[M(0)]^{\frac{1}{2}}$ asymptotically as $k \to \infty$, for all monotonically decreasing $M(\zeta)$ having $M(0) > 0$.

Therefore we may rearrange (26) to arrive at the condition

$$\lambda^2 \frac{d(\sigma^2)}{dk^2} \int_0^\infty M(\zeta)\phi^2 d\zeta = \int_0^\infty [\lambda M(\zeta) - k^2]\,\phi^2 d\zeta. \tag{27}$$

The signs of the integrals on either side of (27) are not immediately obvious. However, by multiplying (22*a*) by ϕ, integrating over all $\zeta \geqslant 0$ and using (22*b*), one sees that

$$\int_0^\infty (\lambda M(\zeta) - k^2)\phi^2 d\zeta = \int_0^\infty (\phi')^2 d\zeta > 0. \tag{28}$$

Thus it follows that the integrals on *both* sides of (27) are positive if $\lambda > 0$. But the discrete spectrum for this problem corresponds to the range

$$\lambda \geqslant k^2/M(0) > 0. \tag{29}$$

Consequently, we have shown that

$$\frac{d(\sigma^2)}{d(k^2)} = \frac{\int_0^\infty (\phi')^2 d\zeta}{\lambda \int_0^\infty [(\phi')^2 + k^2\phi^2]\, d\zeta} > 0 \tag{30}$$

for the discrete spectrum. Since we may take σ and k both positive, (30) shows that

$$d\sigma/dk > 0, \tag{31}$$

so that σ increases with increasing k. By similar arguments, one can also show that λ increases with increasing k, or

$$\frac{d}{dk}\left(\frac{k}{\sigma}\right) > 0. \tag{32}$$

Coupled with (31), (32) implies that

$$\frac{d}{d\sigma}\left(\frac{k}{\sigma}\right) = \frac{1}{\sigma}\left(\frac{dk}{d\sigma} - \frac{k}{\sigma}\right) > 0,$$

or

$$d\sigma/dk < \sigma/k \tag{33}$$

at every value of k. Thus we also see that the $\sigma(k)$ curve is concave downwards at each point and has the general shape shown in figure 2.

3.5. *The maximum value of* $\sigma^2 = M(0)$

Equation (22a) may be written as

$$\phi_{\zeta\zeta} + \lambda[M(\zeta) - \sigma^2]\phi = 0. \tag{34}$$

In intervals for which $\sigma^2 > M(\zeta)$ the solution ϕ has exponential behaviour. If σ^2 exceeds $M(0)$, then ϕ has exponential behaviour for all ζ, and no bounded solution is possible. Therefore, assuming that $M(0) > 0$, the growth rate cannot exceed $[M(0)]^{\frac{1}{2}}$, i.e.

$$\sigma < [M(0)]^{\frac{1}{2}}. \tag{35}$$

We can also show that this is, in fact, attained asymptotically, i.e. $\sigma \to [M(0)]^{\frac{1}{2}}$ as $k \to \infty$, so that

$$\sigma \to \sigma_{\max} = [M(0)]^{\frac{1}{2}} \quad \text{as} \quad k \to \infty. \tag{36}$$

To see this, we note from (32) and (35) that $\lambda \to \infty$ as $k \to \infty$. We may therefore calculate the eigenfunction of (34) as $k \to \infty$ by exploiting the behaviour of λ for large k by using the WBKJ approximation. Write

$$g(\zeta; \sigma) = M(\zeta) - \sigma^2$$

so that (34) becomes

$$\phi_{\zeta\zeta} + \lambda g(\zeta; \sigma)\phi = 0.$$

The zero of $g(\zeta; \sigma)$ occurs at $\zeta = \xi$, where $\xi = M^{-1}(\sigma^2)$ and $\xi \to 0$ as $\lambda \to \infty$. Near $\zeta = \xi$,

$$g(\zeta; \xi) \sim M'(\xi)(\zeta - \xi) = -|M'(\xi)|(\zeta - \xi)$$

since, by hypothesis, $M'(\xi) < 0$. Let $\zeta - \xi = \lambda^{-\frac{1}{3}}X$. Then, for X fixed and of order one,

$$\phi = C\,\mathrm{Ai}(|M'(\xi)|^{\frac{1}{3}}X) \tag{37a}$$

as $\lambda \to \infty$, where Ai is the Airy function, while for fixed $\zeta > \xi$,

$$\phi \sim D\{|M'(\xi)|(\zeta - \xi)^{-\frac{1}{4}} \exp\left\{-\lambda^{\frac{1}{2}} \int_{\xi}^{\zeta} |M(\zeta) - \sigma^2|^{\frac{1}{2}} d\zeta\right\}. \tag{37b}$$

The asymptotic representations match, in the sense of matched asymptotic expansions, if C and D are chosen to satisfy,

$$C = 2\pi^{\frac{1}{2}}\lambda^{\frac{1}{12}}D.$$

The boundary condition at $\zeta = \infty$ is satisfied by (37b). The boundary condition at $\zeta = 0$ corresponds to

$$X = -\lambda^{\frac{1}{3}}\xi.$$

Provided that ξ is $O(\lambda^{-\frac{1}{3}})$, (37a) is a valid asymptotic representation of ϕ right up to the boundary $\zeta = 0$. Assuming that ξ is so chosen, the boundary condition at $\zeta = 0$ is satisfied if

$$-\xi(\lambda|M'(\xi)|)^{\frac{1}{3}} = a_n,$$

where a_n is the nth zero of the Airy function of negative argument. Thus the eigenvalues of (34) are given by

$$\lambda = \lambda_n = -a_n^3/(\xi^3|M'(\xi)|), \quad n = 1, 2, 3, \ldots, \tag{38}$$

and the corresponding eigenfunctions are given approximately by (37) in appropriate ζ intervals.

For example, the first zero (cf. Abramowitz & Stegun 1964, p. 478) $-a_1 = 2{\cdot}338$ corresponds to the lowest eigenfunction and, since $\lambda_n = k^2/\sigma_n^2$, this consequently corresponds to the most unstable mode. Note that the eigenfunction $\phi_1(\zeta)$ vanishes only at the end points $\zeta = 0$ and $\zeta = \infty$, and has no other zeros. The eigenvalues λ_n can be ordered as

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots$$

and there are an infinite number of them, with no finite upper bound. The correspondence between the number of zeros in $\zeta > 0$ and the indices of the eigenvalues follows from Sturm's oscillation theorems (Ince 1956, pp. 232–233) in the usual way.

The analysis above therefore shows that a solution to the eigenvalue problem exists for all $\sigma^2 < M(0)$ and therefore $\sigma \to \sigma_{\max}$ as $k \to \infty$. Thus $M(0) > 0$ is also a sufficient condition for instability (see also § 3.7).

3.6. *Asymptotic behaviour of the eigenvalues λ_n as $n \to \infty$*

For fixed k, $\lambda_n \to \infty$ as $n \to \infty$, which implies that $\sigma_n \to 0$ as $n \to \infty$. This limit is not of central importance to the stability problem, but we nevertheless report the result. Since $\lambda_n \to \infty$, we may again apply the WKBJ method. Now the turning point ξ is not near the left boundary $\zeta = 0$, so we need an asymptotic representation for ϕ_n for fixed $\zeta < \xi$. The WKBJ approximation for $\zeta < \xi$ satisfying the boundary condition at $\zeta = 0$ is

$$\phi \sim B|M'(\xi)|^{-\frac{1}{4}}(\xi-\zeta)^{-\frac{1}{4}}\sin\left\{\lambda_n^{\frac{1}{2}}\int_0^{\zeta}(M(\zeta)-\sigma_n^2)^{\frac{1}{2}}\,d\zeta\right\}. \tag{39}$$

This can be matched to the Airy function solution (see above) in the transition region provided that

$$\lambda_n = \frac{(2n-\frac{1}{4})\pi}{\displaystyle\int_0^{\xi}(M(\zeta)-\sigma_n^2)^{\frac{1}{2}}\,d\zeta}. \tag{40}$$

Since $\lambda_n \to \infty$, $\sigma_n \to 0$ and can be neglected in the denominator of (40). (Then $\xi \to \zeta_0$, the zero of $M(\zeta)$.)

3.7. $\lim_{k\to 0} \sigma_n(k) = 0$ *for the discrete spectrum, unless* $\gamma = 0$

Next we show that a solution to the eigenvalue problem exists with finite values of λ_n as $k \to 0$. Thus, in particular, a finite value of λ_1 exists for $k \to 0$, which, since $\lambda_1 = k/\sigma_1$, implies that $\sigma_1 \to 0$ as $k \to 0$. If $\gamma > 0$, this result holds for all M of the kind considered which have $M(0) > 0$. If $\gamma = 0$, a solution to the eigenvalue problem may not exist for $k = 0$, as we shall see. We note that a consequence of this subsection is that $M(0) > 0$ is a sufficient condition for instability.

To see the result when $\gamma > 0$, formally set $k = 0$ in (22*a*). A turning point still exists, now at the zero ζ_0 of the function $M(\zeta)$. Therefore the construction of §3.6 still applies for the higher eigenvalues ($n \to \infty$). Thus the eigenvalue problem continues to have a solution, and there is a least eigenvalue $\lambda_1 > 0$. Thus for all modes $\sigma_n \to 0$ as $k \to 0$ and therefore the $\sigma_n(k)$ take all values from (and including) $\sigma = 0$ up to (but not including) $\sigma = \sigma_{\max}$ as k ranges from $k = 0$ to $k = \infty$.

When $\gamma = 0$, there is no finite zero of $M(\zeta)$. In this case $M(\zeta) \to 0$ as $\zeta \to \infty$, and if

$$\int_{\zeta_1}^{\infty} \zeta^2 M(\zeta)\, d\zeta < \infty \tag{41}$$

for some number ζ_1, the only solution to the problem is the trivial solution $\phi \equiv 0$ (see Leibovich 1970, pp. 815–816). Condition (41) usually holds, since in this case $M(\zeta) = U_s' U'$, and U_s' (at least) vanishes exponentially fast.

3.8. *Behaviour of the eigenfunctions of the discrete spectrum*

For $\zeta > \xi > \zeta_0$, where ξ is the zero of $M(\zeta) - \sigma^2$ and ζ_0 is the zero of M, the eigenfunctions display exponential behaviour, while for $\zeta < \xi$ they display sinusoidal behaviour. It can be shown that the last local maximum (or minimum) of $\phi_n(\zeta)$ occurs at a point ζ_m such that $\zeta_m < \xi < \zeta_0$. This may be seen by multiplying (34) by ϕ_ζ and integrating from $\zeta = \zeta_m$ to $\zeta = \infty$. The result is

$$[M(\zeta_m) - \sigma_n^2]\, \phi^2(\zeta_m) = -\int_{\zeta_m}^{\infty} M'(\zeta)\, \phi^2 d\zeta. \tag{42}$$

By hypothesis $M'(\zeta) \leqslant 0$, and for the discrete spectrum $\lambda_n > 0$. Therefore

$$M(\zeta_m) - \sigma_n^2 > 0.$$

Since M is a decreasing function and $M(\xi) - \sigma_n^2 = 0$, $\zeta_m < \xi$. Therefore the magnitude of the eigenfunctions decreases exponentially for $\zeta > \zeta_m$. In particular, the eigenfunction corresponding to the lowest mode ceases to be oscillatory and reaches its final maximum value for $\zeta < \xi$.

Notice that for fixed U_s and U this result shows that the effective depth of the unstable motion decreases with increasing γ. Since M decreases as γ increases, ζ_0 decreases as γ increases. Therefore the value of γ controls the penetration depth of the instability

The asymptotic behaviour of the eigenfunctions is given by

$$\phi_n(\zeta) \sim \text{constant} \times \exp[-(\lambda_n \gamma + k^2)^{\frac{1}{2}} \zeta] \quad \text{as} \quad \zeta \to \infty. \tag{43}$$

3.9. *The continuous spectrum*

If $\lambda = k^2/\sigma^2 = -\mu^2 < 0$, then (22*a*) becomes

$$\phi_{\zeta\zeta} - [\mu^2 M(\zeta) + k^2]\phi = 0. \tag{44}$$

If $\mu^2 M(\zeta) + k^2 > 0$ everywhere, then (44) has no bounded solutions. If $\mu^2 M + k^2$ vanishes at $\zeta = \xi$, then the solutions of (44) will be of exponential type for $\zeta < \xi$ and of sinusoidal type for $\zeta > \xi$. In the latter case, ϕ will be bounded as $\zeta \to \infty$, and will have the asymptotic behaviour

$$\phi \sim \text{constant} \times \sin\{(\mu^2\gamma - k^2)^{\frac{1}{2}}\zeta + \alpha\}$$

for some α. These eigenfunctions are improper (they are not square integrable and do not satisfy the boundary condition at $\zeta = \infty$), and correspond to internal waves propagating to infinite depths. The spectrum is continuous for all $\mu^2 > k^2/\gamma$.

4. Examples

Two simple examples, similar to those of Craik (1977), illustrate the general results of §3.

(*a*) Let

$$M(\zeta) = \begin{cases} m_0^2 > 0, & \zeta < L, \\ -\gamma, & \zeta > L, \end{cases}$$

where m_0^2 is a constant. This corresponds to a linear U and U_s. The eigenfunctions are

$$\phi_n = \begin{cases} A\sin[\zeta(\lambda_n m_0^2 - k^2)^{\frac{1}{2}}] & \text{for } \zeta < L, \\ A\sin[L(\lambda_n m_0^2 - k^2)^{\frac{1}{2}}]\exp\{-(\zeta - L)(\lambda_n\gamma + k^2)^{\frac{1}{2}}\} & \text{for } \zeta > L, \end{cases}$$

and the eigenvalues are determined by the dispersion relation

$$\tan p = -\frac{m_0}{\gamma^{\frac{1}{2}}}\frac{p}{(p^2 + \beta^2)^{\frac{1}{2}}},$$

where

$$p = L(\lambda_n m_0^2 - k^2)^{\frac{1}{2}}, \quad \beta^2 = k^2L^2(1 + m_0^2/\gamma).$$

Let p_∞ satisfy the equation

$$\tan p_\infty = -m_0/\gamma^{\frac{1}{2}}$$

or

$$p_\infty = \tfrac{1}{2}\pi(1+\delta) + \nu\pi, \quad \nu = 0, 1, 2, \ldots,$$

where δ is a well-determined number that depends upon $m_0/\gamma^{\frac{1}{2}}$ and lies in the interval $0 < \delta \leqslant 1$. Denote the value of p corresponding to the nth mode by p_n; then $n = \nu + 1$ and

$$p_\infty < p_n \leqslant (\nu + 1)\pi,$$

or

$$\frac{\pi^2[\frac{1}{2}(1+\delta) + \nu]^2 + k^2L^2}{m_0^2L^2} < \lambda_n \leqslant \frac{(\nu+1)^2\pi^2 + k^2L^2}{m_0^2L^2}.$$

All features described in §3 may be easily verified. For example, as $k \to \infty$, $\beta \to \infty$ and therefore $\tan p \to 0$ for any mode with $n < \infty$; hence $\delta \to 1$ and

$$\lambda_n = k^2/\sigma_n^2 \sim k^2 m_0^{-2}(1 + O(\nu^2/L^2k^2)),$$

or

$$\sigma_n^2 \sim m_0^2 = M(0),$$

as $k \to \infty$ in agreement with §3.5. Similarly, as $k \to 0$, $\beta \to 0$, and therefore

$$p = Lm_0\lambda_n^{\frac{1}{2}} = p_\infty = \text{constant}$$

so

$$\sigma_n \sim \frac{Lm_0 k}{p_\infty} = \frac{2Lm_0}{\pi(2n + \delta - 1)}k,$$

in agreement with §3.7.

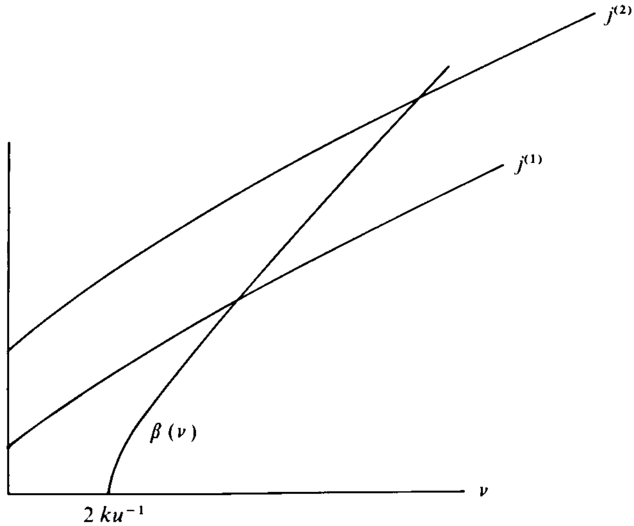

FIGURE 3. Construction leading to $\sigma(k)$ for example (b) of §4. The parameter ν_n is given by the intersection of the $\beta(\nu)$ curve with $j_\nu^{(n)}$. The growth rate σ_n of the nth eigenmode may be found from ν_n.

(b) As a second example, take the more realistic case

$$M(\zeta) = M_0^2\mu^2 \exp(-\mu\zeta) - \gamma.$$

This corresponds to a situation in which the product $U_s'U'$ decays exponentially, as might be expected, and the water density is linearly stratified.

The change of variable $y = e^{-\mu\zeta}$ transforms the eigenvalue equation into a Bessel equation, and the solution satisfying the boundary conditions is

$$\phi = CJ_\nu(\beta \exp[-\tfrac{1}{2}\mu\zeta]),$$

where J_ν is the Bessel function of order $\nu > 0$ and

$$\beta \equiv 2M_0\lambda_n^{\frac{1}{2}} = j_\nu^{(n)}, \quad \nu = (2/\mu)[k^2 + \gamma\lambda_n]^{\frac{1}{2}}. \tag{45}$$

Here $j_\nu^{(n)}$ is the nth zero of J_ν.

We note that the asymptotic behaviour of j_ν as $\nu \to \infty$ is (Abramowitz & Stegun 1964, §5.22, p. 371)

$$j_\nu^{(n)} \sim \nu + b_n \nu^{\frac{1}{3}} + O(\nu^{-\frac{1}{3}}),$$

where b_n depends upon n. From the definitions of β and ν, we see that, on eliminating λ_n, β can be expressed as a function of ν alone,

$$\beta(\nu) = (\mu M_0/\gamma^{\frac{1}{2}})(\nu^2 - 4k^2\mu^{-2})^{\frac{1}{2}},$$

and that the asymptotic slope of the $\beta(\nu)$ curve as $\nu \to \infty$ is $\mu M_0/\gamma^{\frac{1}{2}}$. The schematic construction in figure 3 then shows how the dispersion relation (45) may be solved.

If $\mu M_0 > \gamma^{\frac{1}{2}}$, then the $\beta(\nu)$ curve will intersect all the $j_\nu^{(n)}$ curves at points $\nu = \nu_n(k)$ that depend upon the value of k/μ. Note that this slope condition is equivalent to the condition $M(0) > 0$, which is the necessary and sufficient condition for instability. From (45), the eigenvalues are

$$\sigma_n = 2M_0 k/\nu_n(k).$$

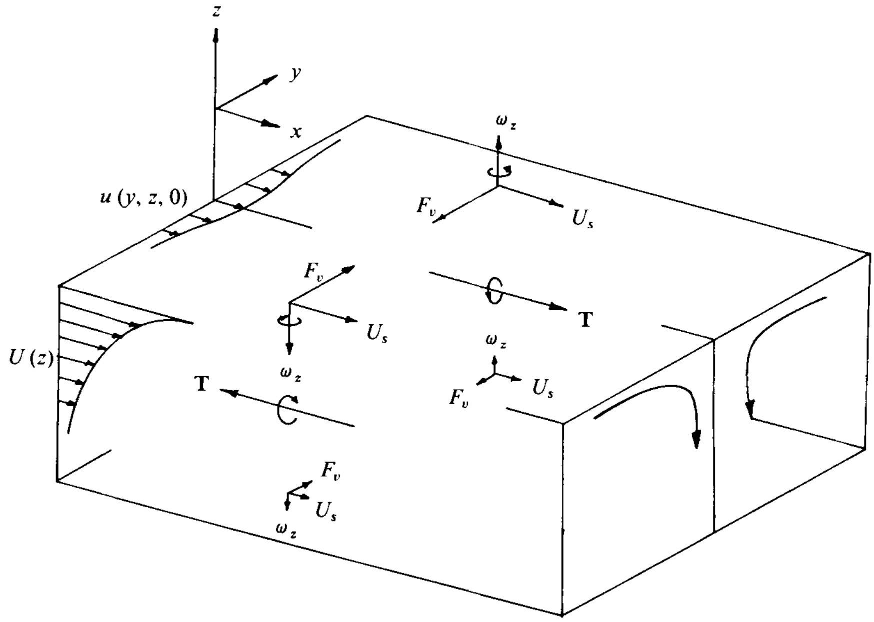

FIGURE 4. Sketch explaining the physical origin of the instability in terms of torques produced by variations of the vortex force. Only the vortex-force component (F_v) arising from the x-independent velocity perturbation $u(y,z,0)$ is shown.

Notice that the value of $\nu_n(0)$ is well determined and finite, so that as $k \to 0$

$$\sigma_n = 2M_0 k/\nu_n(0),$$

in conformity with the general results in §3.7. Similarly, as $k \to \infty$, $\nu_n \to \infty$ (since $\nu_n > 2k/\mu$). Since $j_\nu^{(n)} \sim \nu$ as $\nu \to \infty$, (45) has the solution

$$\nu_n^2 \sim 4k M_0^2/(\mu^2 M_0^2 - \gamma),$$

which gives $$\sigma_n^2 \sim \mu^2 M_0^2 - \gamma = M(0),$$

which is in conformity with §3.5.

5. Physical explanation of the instability: analogy with thermal convection

Craik (1977) has given an explanation of the instability (in constant density water) in terms of the vorticity. Here we prefer to base our discussion on the 'vortex force' $\mathbf{u}_s \times \langle \boldsymbol{\omega} \rangle$ imposed on the mean flow by the simultaneous presence of waves (hence $\mathbf{u}_s$) and currents (hence $\langle \boldsymbol{\omega} \rangle \neq 0$).

Suppose that a parallel shear flow has developed, so that $\langle \mathbf{v} \rangle = U\mathbf{i}$, with $\partial U/\partial z \neq 0$, and that surface waves impose a Stokes drift $\mathbf{u}_s = U_s(z)\mathbf{i}$, with $U_s'(z) > 0$. Also, let $\langle \theta \rangle = 0$. This is a possible state of motion, and satisfies (12) (if $\nu_T \neq 0$, then U must be time dependent, and $U_t = \nu_T U_{zz}$). The vorticity associated with the motion is

$$\langle \boldsymbol{\omega} \rangle = (-\partial U/\partial z)\mathbf{j}$$

(where $\mathbf{j}$ is a unit vector in the y direction) and is perpendicular to $\mathbf{u}_s$, so the vortex force is vertically upward and uniform in horizontal planes. This force is felt as a pressure, balanced by π, and induces no accelerations.

Now suppose that the velocity profile is slightly disturbed at time $t = 0$ by an amount $u(y,z,0)$ as shown in figure 4. Then the perturbation is y dependent, and is associated with a vertical vorticity component $-\partial\langle u\rangle/\partial y$. In the disturbance illustrated in figure 4, the vertical component of vorticity ω_z is negative for $y < 0$ and positive for $y > 0$. The displacement does not affect $\mathbf{u}_s(z)$, so a vortex force acts on either side in the direction of the plane $y = 0$, where $u(y,z,0)$ has its maximum. Since $\mathbf{u}_s$ decreases with depth, a similar but weaker set of vortex forces will act at greater depth. The vortex-force distribution with depth on a plane $y = y_0 =$ constant therefore produces a net moment $\mathbf{T}$ in the direction of x increasing if $y_0 > 0$ or in the opposite direction if $y_0 < 0$. The directions of the couples, shown in figure 4, are such as to produce a circulation with a convergence at the plane $y = 0$, where $u(y,z,0)$ has its maximum. Therefore the vertical velocity component is negative at $y = 0$, and the non-diffusive form of (12) requires that the perturbation u satisfies

$$\partial u/\partial t \doteqdot -\langle w\rangle \partial U/\partial z$$

at a plane of downwelling (since $\langle v\rangle = 0$ there and $U \gg u$) in an x-independent motion. Since $\langle w\rangle < 0$ and $\partial U/\partial z > 0$ at $y = 0$, $\partial u/\partial t > 0$. Therefore the initial perturbation $u(y,z,0)$ grows with time.

The discussion so far has ignored density effects in order to focus on the destabilizing forces. The tendency for downwelling to form leads to a zone of light fluid near $y = 0$ and heavier fluid on either side. Thus the gravity forces, which are vertical, are greater away from the plane $y = 0$ than they are on that plane. Consequently, the gravity force on a plane $z =$ constant produces a torque on the fluid that always opposes the torque $\mathbf{T}$ described above. Therefore stable stratification opposes the tendency to overturn the fluid, and stability or instability results from the net action of these opposing forces.

The driving mechanism in CL and I may be explained in a similar way, but it is the imposed spanwise variation of u_s that produces a net torque (even without an initial perturbation of the vortex lines).

Next observe that for x-independent motion (18) show that, if $U_s' =$ constant, the effect of the vortex force is completely analogous to that of the buoyancy force, and the x component of the velocity is analogous to temperature. Although (18) have been linearized, their nonlinear counterparts share this characteristic. Now $U_s' =$ constant is not usually a justifiable approximation. Nevertheless, if one makes this assumption, results in the thermal-convection literature may, when suitably interpreted, be applied to the present problem.

Consider the question of the infinitesimal steady convection, with heat conduction and viscosity accounted for but with $U_s'(z)$ taken to be a constant. From (18), this problem is governed by (after suitable reduction)

$$\nu_T^2 \Delta^3 w = (U_s' U' - \beta g \overline{T}' Pr_T)\Delta_1 w$$

(where $Pr_T = \nu_T/\alpha_T$). Let us choose κ^{-1} to be our length scale, where κ is a wavenumber characterizing the surface waves. If a typical amplitude of the surface waves is ϵ/κ and the corresponding frequency is σ_s, then the Stokes-drift gradient has a scale $2\epsilon^2\sigma_s$. At the surface, the stress condition requires that $U' = u_*^2/\nu_T$, where u_* is the water friction velocity, so we take u_*^2/ν_T as a scale for U' and set

$$U' U_s' = (2\epsilon^2 \sigma_s U_*^2/\nu_T) G(\zeta),$$

where G is a dimensionless function of $\zeta = -\kappa z$.

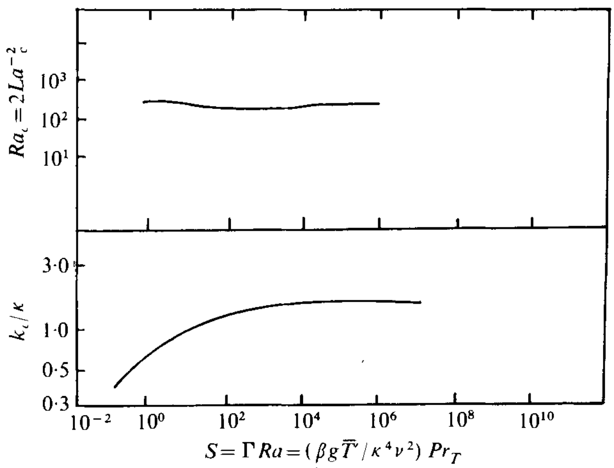

FIGURE 5. The critical 'Rayleigh' number (also twice La^{-2}, where La is the Langmuir number from I) and critical wavenumber k_c found by Whitehead & Chen are shown as functions of S. Here S is the ratio of the stabilizing force due to buoyancy to the destabilizing vortex force.

Set

$$Ra = \frac{2\epsilon^2\sigma_s}{(\kappa\nu_T)^3}u_*^2, \quad \Gamma = Pr_T\frac{\nu_T\beta g\overline{T}'}{2\epsilon^2\sigma_s u_*^2},$$

$$S = \Gamma Ra, \quad f(\zeta) = G(\zeta) - \Gamma.$$

Then

$$\Delta^2 w = Ra f(\zeta)\Delta_1 w.$$

This equation is of the form treated by Whitehead & Chen (1970) and we have introduced the same symbol as they employ for the parameter Ra (their Rayleigh number). We note that, in the notation of I,

$$Ra = 2La^{-2},$$

where La is the Langmuir number. Whitehead & Chen treat several examples of functions $f(\zeta)$. In particular, they consider the example

$$f = \mu^2 M_0^2 \exp(-\mu\zeta) - \Gamma,$$

with Γ = constant, which is considered in §4. For a stress-free isothermal top boundary ($w = w_{zz} = \theta = 0$ at $z = 0$), they found that the critical Rayleigh number Ra and critical wavenumber k_c varied with the parameter S as shown in figure 5 (adapted from their figure 4).

For the example illustrated, the critical Rayleigh (or Langmuir) number is insensitive to the stabilizing gradient S and to a lesser extent the same is true for the critical wavenumber. For other examples calculated by Whitehead & Chen the critical wavenumber (in particular) showed a greater dependence on S.

Although the critical Rayleigh number and wavenumber depend only weakly upon S, the plots of eigenfunctions in Whitehead & Chen's figure 5 show that the effective depth of penetration (and hence the cell aspect ratio) does depend strongly on S. The effective depth of the unstable layer decreases with increasing stabilizing gradient S, which is a behaviour that is apparent from the inviscid analysis of §3.8.

6. Discussion: application to Langmuir circulations

The inviscid stability analysis tells us that growth rates are a fraction of

$$\sigma_{\max} = (U_s'(0)\,U'(0) - \beta g \overline{T}'(0))^{\frac{1}{2}}$$

and that the most unstable wave is that of vanishingly small wavelength. The Brunt–Väisälä frequency $N(z) = (\beta g \overline{T}'(z))^{\frac{1}{2}}$, and $U_s'(0)$ and $U'(0)$ can be characterized as in the last section, so we may write $\sigma_{\max}$ as

$$\sigma_{\max} \sim (2\epsilon^2 \sigma_s u_*^2/\nu_T - N^2(0))^{\frac{1}{2}}. \quad (46)$$

N^2 is usually very small, particularly near the surface, and can be neglected. One way to estimate $\sigma_{\max}$ is to refer to I [equation (14)], which shows that $u_*^2/\nu_T \sigma_s$ is of the order of the wind-drift current (with a scale measured in I by $\epsilon_c = u_*^2/\nu_T \sigma_s$). Since the wind-drift current is of the order of 3 % of the wind speed, we estimate (46) to be

$$\sigma_{\max} \sim \epsilon \sigma_s (0{\cdot}06)^{\frac{1}{2}} = 0{\cdot}2 \epsilon \sigma_s.$$

A typical value for ϵ is $\frac{1}{8}$, so we may estimate

$$\sigma_{\max} = O(0{\cdot}025\,\sigma_s).$$

The frequency σ_s is typically of the order of $1\ \mathrm{s}^{-1}$, which gives an e-folding time of 40 s.

A growth factor of 10^4, say, requires a growth time of 6 min for the most unstable waves on the basis of the estimates of the previous paragraph. However, the effects of viscosity are certainly not ignorable for waves with wavenumbers as large as, or larger than, $(\sigma_{\max}/\nu_T)^{\frac{1}{2}}$. The high wavenumbers, which are most unstable on inviscid grounds, are certainly damped by the dissipative effects that we have left out of the analysis. (For the value of $\sigma_{\max}$ found above and $\nu_T = 10\ \mathrm{cm}^2/\mathrm{s}$, the viscous cut-off wavenumber is $(\frac{1}{20})\ \mathrm{cm}^{-1}$, corresponding to a disturbance wavelength of 1·26 m.) Consequently a somewhat smaller growth rate, and thus a longer growth time, is indicated. Nevertheless, one expects a growth time that is a modest multiple of the estimate above, which would then be in accord with observed Langmuir-circulation growth times.

As indicated above, there is no preferred wavelength in the non-dissipative problem. When viscosity and heat conduction are allowed, however, as in the (imperfect) analogy in the previous section, a preferred wavelength does emerge. For the example described in §5, the preferred wavenumber depends upon the strength of the stable density gradient, but varies from small values up to about $k = 2\kappa$. (We note that we may write $S = N^2 Pr\, \kappa^{-4} \nu_T^{-2}$. For $\kappa \sim 1\ \mathrm{m}^{-1}$, $N \sim 5 \times 10^{-3}\ \mathrm{s}^{-1}$ and $\nu_T = 20\ \mathrm{cm}^2/\mathrm{s}$, $S \sim 6{\cdot}25\, Pr$. For $Pr \sim 7$, $S = 44$, where, from §5, $k_c/\kappa \doteqdot 1$. Reducing ν_T by half or more raises S to a range where k_c/κ is level and approximately equal to 2.) Thus the wavelength of the preferred mode is of the order of the wavelength of the dominant surface waves. The critical Rayleigh number for the example cited is about 300. Since $Ra = 2La^{-2}$, and values of La are, according to I (§7), typically of the order of 10^{-3} in the oceans, we expect oceanic conditions to be far above critical.

The example above should be viewed with caution, since the neglected effects of the curvature of the Stokes-drift profile will certainly alter the quantitative details. Consequently, one must not place much weight on the values of the critical wavenumber or Rayleigh number. Nevertheless, it is unlikely that a proper analysis would lead to a critical Rayleigh number several orders of magnitude larger than the one cited in § 5.

Thus we interpret the 'analogy' and the non-dissipative stability analysis as both suggesting that a wide range of typical conditions in the ocean are highly unstable on linear grounds as a result of the rectified effects of surface wave activity. It would, in my view, be unwise to offer a theoretical mechanism as an 'explanation' of Langmuir circulations until the fully nonlinear problem has been carefully explored and compared with observation. The instability mechanism has interesting possibilities as an ultimate explanation, and we have therefore undertaken a numerical study of finite amplitude unstable motions that we hope to report in due course.

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